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A NOTE ON THE GAUGE INVARIANT UNIQUENESS THEOREM FOR C*-CORRESPONDENCES

EVGENIOS T.A. KAKARIADIS

Abstract. We present a short proof of the gauge invariant uniqueness theorem for relative Cuntz-Pimsner algebras of C*-correspondences.

1. Introduction

Soon after its initiation by Pimsner [17], the theory of C*-correspondences captured the interest of the research community. The motivating feature is their flexible language that encodes a broad variety of examples in operator algebras (both selfafdjoint and nonselfadjoint). Nowadays they give a central construction in the general theory of C*-algebras [2], following the general framework provided by Katsura [9, 10, 11]. In this note we present a short proof of the gauge invariant uniqueness theorem for relative Cuntz-Pimsner algebras by using a less sharp analysis of the cores than that of [10].

The terminology of C*-correspondences has been under a number of considerable changes in the last years. In this paper we follow [10]. A *C*-correspondence X over a C*-algebra A* is a right Hilbert A-module along with a $*$-homomorphism $\phi_X\colon A\to\mathcal{L}(X)$ on the adjointable operators $\mathcal{L}(X)$. We say that a pair (π,t) defines a representation of X if $\pi\colon A\to\mathcal{B}(H)$ is a $*$-representation and $t\colon X\to\mathcal{B}(H)$ is a linear map such that $\pi(a)t(\xi)\pi(b)=t(\phi_X(a)\xi b)$ and $t(\xi)^*t(\eta)=\pi(\langle\xi,\eta\rangle_X)$ for all $a,b\in A$ and $\xi,\eta\in X$. The C*-property implies that t is isometric when π is injective. Kajiwara, Pinzari and Watatani [12] show that an (injective) pair (π,t) induces an (injective) $*$-representation $\psi_t\colon\mathcal{K}(X)\to\mathcal{B}(H)$ such that $\psi_t(\theta_{\xi,\eta})=t(\xi)t(\eta)^*$. As we are about to see, the ideal

$$I'_{(\pi,t)}:=\{a\in A\mid \pi(a)\in\psi_t(\mathcal{K}(X))\},$$

plays a significant role (see also Katsura's work [10, 11]). We say that (π,t) *admits a gauge action* $\{\gamma_z\}_{z\in\mathbb{T}}$, if $\{\gamma_z\}_{z\in\mathbb{T}}$ is a point-norm continuous family of $*$-endomorphisms with

$$\gamma_z(\pi(a))=\pi(a),\ \text{for}\ \ a\in A,\quad \gamma_z(t(\xi))=zt(\xi),\ \text{for}\ \ \xi\in X.$$

Let J be an ideal of A contained in $\phi_X^{-1}(\mathcal{K}(X))$. We say that a pair (π,t) is *J-covariant* if $\pi(a)=\psi_t(\phi_X(a))$ for all $a\in J$. Then $\mathcal{O}(J,X)$ is the universal C*-algebra generated by (the copies of) A and X relative to

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J-covariant pairs (π, t). When $J = (0)$ we denote $\mathcal{O}(J, X)$ by $\mathcal{T}_X$ which is called the *Toeplitz-Pimsner algebra*. When J is Katsura's ideal [10]

$$J_X := \ker \phi_X^{\perp} \cap \phi_X^{-1}(\mathcal{K}(X))$$

we write $\mathcal{O}_X \equiv \mathcal{O}(J_X, X)$ which is called the *Cuntz-Pimsner algebra*. In particular $\mathcal{T}_X$ is the universal C*-algebra relative to the representations of X, hence $\mathcal{O}(J, X)$ is the quotient of $\mathcal{T}_X$ by the ideal generated by the elements $\pi(a) - \psi_t(\phi_X(a))$ for all $a \in J$. It should be noted that Pimsner [17] considers C*-algebras generated simply by a copy of X. However the C*-correspondences therein are assumed to be (injective and) *full*, i.e., $\langle X, X\rangle$ is dense in A [17, Remark 1.2 (3)]. That is the reason why the C*-algebras in [17] that are generated simply by X manage to reconstruct a copy of A.

There is a strong connection between $\mathcal{T}_X$ and $\mathcal{O}_X$ attained by Muhly and Solel [15] and Fowler, Muhly and Raeburn [4] under certain assumptions, and settled in full generality by Katsoulis and Kribs [8]. The *tensor algebra* $\mathcal{T}_X^+$ in the sense of Muhly and Solel [15] is the non-involutive closed subalgebra of $\mathcal{T}_X$ generated by A and X. Then $\mathcal{O}_X$ is the minimal C*-cover of $\mathcal{T}_X^+$, i.e., the *C*-envelope* of $\mathcal{T}_X^+$ in the sense of Arveson [1].

A key role in the theory of C*-correspondences is played by the gauge invariant uniqueness theorems. This type of result was initiated by an Huef and Raeburn for Cuntz-Krieger algebras [6, Theorem 2.3] and various generalizations were given by Doplicher, Pinzari and Zuccante [3, Theorem 3.3], Fowler, Muhly and Raeburn [4, Theorem 4.1], and Fowler and Raeburn [5, Theorem 2.1]. In all these cases at least injectivity of ϕ_X is assumed, therefore they were not enough to treat general constructions such as C*-algebras of graphs with sources. Gauge invariant uniqueness theorems for $\mathcal{T}_X$ and $\mathcal{O}_X$ were given in full generality by Katsura [10, Theorem 6.2, Theorem 6.4] by using a sharp analysis of the ideal structure of the cores and a conceptual argument concerning short exact sequences. An alternative proof for $\mathcal{O}_X$ was given afterwards by Muhly and Tomforde [16] by using a tail adding technique. An extended gauge invariant uniqueness theorem in the much broader class of C*-algebras associated to the pre-C*-correspondences is given by Kwaśniewski [13].

The Gauge Invariant Uniqueness Theorem. *Let X be a C*-correspondence over A and let J be an ideal of A contained in J_X. Then a pair (π, t) defines a faithful representation of the J-relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ if and only if (π, t) admits a gauge action, π is injective and $I'_{(\pi,t)} = J$.*

As an immediate consequence of the gauge invariant uniqueness theorem we obtain that if (π, t) admits a gauge action and π is injective then $\mathrm{C}^*(\pi, t) \simeq \mathcal{O}(I'_{(\pi,t)}, X)$. We remark that for Toeplitz-Pimsner algebras the condition $I'_{(\pi,t)} = (0)$ implies injectivity of π, whereas for the Cuntz-Pimsner algebra $\mathcal{O}_X$ the condition $I'_{(\pi,t)} = J_X$ is redundant. Indeed when π is injective then Katsura remarks that $I'_{(\pi,t)} \subseteq J_X$ [10, Proposition 3.3]. Finally, when $J \subseteq \phi_X^{-1}(\mathcal{K}(X))$ then A (and consequently X) embeds isometrically in $\mathcal{O}(J,X)$, if and only if $J \subseteq J_X$, if and only if $\phi_X|_J$ is injective [7, Lemma 2.7]. Therefore Katsura's ideal J_X is the maximal ideal for obtaining the minimal C*-algebra that contains an isometric copy of X.

2. The Proof

Preliminaries. We follow notation and terminology of [10]. We write $X^{\otimes 0} = A$ and $X^{\otimes n+1} = X^{\otimes n} \otimes_A X$ for $n \geq 0$, i.e., *the A-stabilized tensor product*. Every $X^{\otimes n}$ becomes a C*-correspondence over A by $\phi_{X^{\otimes n}} = \phi_X \otimes \mathrm{id}_{n-1}$ when $n \geq 1$, and ϕ_{X^0} is the multiplication action on A. We write $\phi_n \equiv \phi_{X^{\otimes n}}$ for simplicity.

Given a representation (π,t) of a C*-correspondence X that acts on a Hilbert space H, let $\mathrm{C}^*(\pi,t)$ be the C*-subalgebra of $\mathcal{B}(H)$ generated by $\pi(A)$ and $t(X)$. We denote by (π,t^n) the induced pair on $X^{\otimes n}$ such that $t^n(\xi_1 \otimes \cdots \otimes \xi_n) = t(\xi_1)\dots t(\xi_n)$ for all $\xi_1,\dots,\xi_n \in X$. Then

$$\mathrm{C}^*(\pi,t) = \overline{\mathrm{span}}\{t^n(\overline{\xi})t^m(\overline{\eta})^* \mid \overline{\xi} \in X^{\otimes n}, \overline{\eta} \in X^{\otimes m}, n,m \in \mathbb{Z}_+\}.$$

When π is injective then the equation $\pi(a) = \psi_t(k)$ implies that $\phi_X(a) = k$ and $a \in J_X$ [10, Proposition 3.3]. In short, the C*-identity implies that t is injective when π is injective, therefore

$$\begin{aligned}\|\phi_X(a)\xi - k\xi\|_X &= \|t(\phi_X(a)\xi) - t(k\xi)\|_{\mathcal{B}(H)} \\ &= \|(\pi(a) - \psi_t(k))t(\xi)\|_{\mathcal{B}(H)} = 0,\end{aligned}$$

for all $\xi \in X$, which shows that $\phi_X(a) = k$. Moreover, since $\pi(b)t(\xi)t(\eta)^* = t(\phi_X(b)\xi)t(\eta)^*$, then for $b \in \ker\phi_X$ we obtain

$$\pi(ba) = \pi(b)\psi_t(k) = \psi_t(\phi_X(b)k) = 0,$$

which shows that $a \in \ker\phi_X^\perp$ due to the injectivity of π.

Let the *cores* of $\mathrm{C}^*(\pi,t)$ be the C*-subalgebras

$$B_{[l,m]} = \mathrm{span}\{\psi_{t^n}(k_n) \mid k_n \in \mathcal{K}(X^{\otimes n}), l \leq n \leq m\}.$$

To see that the $*$-subalgebras $B_{[l,m]}$ are indeed closed, first note that $B_{[l,l]}$ is closed since ψ_{t^l} has closed range. Moreover $\psi_{t^{l+1}}(\mathcal{K}(X^{\otimes l+1}))$ is a closed ideal of $B_{[l,l+1]}$ hence

$$B_{[l,l+1]} = B_{[l,l]} + B_{[l+1,l+1]} = q^{-1} \circ q(B_{[l,l]})$$

is closed, where $q \colon \overline{B_{[l,l+1]}} \to \overline{B_{[l,l+1]}}/B_{[l+1,l+1]}$ is the usual quotient $*$-epimorphism. Inductively we get that $B_{[l,m]}$ is a C*-subalgebra of $\mathrm{C}^*(\pi,t)$.

If (e_i) is an approximate identity of $\mathcal{K}(X)$ then $(\psi_t(e_i))$ is an approximate identity of $B_{[n,n]}$ for all $n \geq 1$, since $\psi_{t^n}(\mathcal{K}(X^{\otimes n}))$ is the closure of the linear span of $t(\xi_1)\dots t(\xi_n)t(\eta_n)^*\dots t(\eta_1)^*$ for $\xi_i,\eta_i \in X$. Consequently $(\psi_t(e_i))$ is an approximate identity for $B_{[1,m]}$ for all $m \geq 1$. Furthermore

$$t(X)^* \cdot B_{[1,m]} \cdot t(X) \subseteq B_{[0,m-1]}, \text{ for all } m \in \mathbb{Z}_+.$$

Furthermore a gauge action $\{\gamma_z\}_{z\in\mathbb{T}}$ on a pair (π,t) defines the conditional expectation

$$E(f) = \int_z \gamma_z(f)dz, \text{ for all } f \in \mathrm{C}^*(\pi,t), \text{ for all } f \in \mathrm{C}^*(\pi,t).$$

Then the fixed point algebra $E(\mathrm{C}^*(\pi,t)) = \mathrm{C}^*(\pi,t)^\gamma$ is the inductive limit of the C*-subalgebras $B_{[0,n]}$.

The Fock representation. For $\xi \in X$ let $\tau_n(\xi) \in \mathcal{L}(X^{\otimes n}, X^{\otimes n+1})$ such that $\tau_n(\xi)(\eta_1\otimes\cdots\otimes\eta_n) = \xi\otimes\eta_1\otimes\cdots\otimes\eta_n$, for $\eta_1,\dots,\eta_n \in X$. Let the *Fock space* $\mathcal{F}(X) = \oplus_{n\in\mathbb{Z}_+} X^{\otimes n}$ be the direct sum Hilbert A-module of $X^{\otimes n}$. The *full Fock representation* is then defined by (π,t) with

$$\pi(a) = \sum_{n\geq 0}\phi_n(a), \text{ for } \ a\in A, \quad \text{ and } \quad t(\xi) = \sum_{n\geq 0}\tau_n(\xi), \text{ for } \ \xi \in X.$$

Then (π,t) defines a representation of X. A useful fact is that $\psi_t\colon \mathcal{K}(X) \to \mathcal{L}(\mathcal{F}(X))$ takes up the form

$$\psi_t(k) = \sum_{n\geq 1} k\otimes \mathrm{id}_{n-1}, \text{ for all } k \in \mathcal{K}(X).$$

Indeed it suffices to note that if $k = \theta_{\xi,\eta}$, then $\tau_n(\xi)\tau_n(\eta)^* = \theta_{\xi,\eta}\otimes\mathrm{id}_{n-1} = k\otimes\mathrm{id}_{n-1}$. Therefore for $a\in A$ such that $\phi_X(a) = k \in \mathcal{K}(X)$ we obtain

$$\pi(a) - \psi_t(\phi(a)) = \phi_0(a) + \sum_{n\geq 1}(\phi_n(a) - k\otimes\mathrm{id}_{n-1}) = \phi_0(a).$$

Moreover (π,t) admits a gauge action $\{\gamma_z\}_{z\in\mathbb{T}}$ by letting $\gamma_z = \mathrm{ad}_{u_z}$ where

$$u_z(\xi_1\otimes\cdots\otimes\xi_n) := z^n\xi_1\otimes\cdots\otimes\xi_n,$$

defines an adjointable unitary operator in $\mathcal{L}(\mathcal{F}(X))$.

Let J be an ideal of A contained in $\phi_X^{-1}(\mathcal{K}(X))$. Then

$$\mathcal{K}(XJ) = \overline{\mathrm{span}}\{\theta_{\xi a,\eta} \mid \xi,\eta\in X, a\in J\},$$

is a closed ideal in $\mathcal{K}(X)$. A crucial remark is that if $\phi_X(a)\in\mathcal{K}(X)$ then $\phi_X(a)\in\mathcal{K}(XJ)$ if and only if $\langle \eta, \phi_X(a)\xi\rangle_X \in J$, which follows by [4, Lemma 2.6] or [11, Lemma 1.6]. In short, if $\langle k\xi, k\xi\rangle_X \in J$ then there is a $\xi'\in X$ and a positive $a\in J$ such that $k\xi = \xi' a$ by [14, Lemma 4.4]. Then

$$k\theta_{\xi,\eta} = \theta_{\xi' a,\eta} = \theta_{\xi'\sqrt{a},\eta\sqrt{a}} \in \mathcal{K}(XJ).$$

Since $k\in\mathcal{K}(X)$ then $k = \lim_i ke_i$ for some approximate identity (e_i) in $\mathcal{K}(X)$ and the above remark shows that the convergent net (ke_i) is in the closed ideal $\mathcal{K}(XJ)$, thus $k\in\mathcal{K}(XJ)$. Conversely if k is the norm limit of some $k_i = \sum_{m=0}^{N_i}\theta_{\xi_m a_m,\eta_m} \in \mathcal{K}(XJ)$ then

$$\langle \eta, \theta_{\xi_m a_m,\eta_m}\xi\rangle_X = \langle \eta,\xi_m\rangle_X\, a_m\, \langle \eta_m,\xi\rangle_X \in J,$$

which implies that $\langle \eta, \phi_X(a)\xi\rangle_X \in J$.

Let the quotient $*$-epimorphism $q_J \colon \mathcal{L}(\mathcal{F}(X)) \to \mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X)J)$ where

$$\mathcal{K}(\mathcal{F}(X)J) = \overline{\operatorname{span}}\{\theta_{\overline{\xi}a,\overline{\eta}} \mid \overline{\xi}, \overline{\eta} \in \mathcal{F}(X), a \in J\}.$$

Then $(q_J \circ \pi, q_J \circ t)$ is a well defined J-covariant representation, since

$$\pi(a) - \psi_t(\phi(a)) = \phi_0(a) \in J \subseteq \mathcal{K}(\mathcal{F}(X)J),$$

for all $a \in J$. In particular note that $\psi_{q_J \circ t} = q_J \circ \psi_t$ on $\mathcal{K}(X)$. Furthermore $\mathcal{K}(\mathcal{F}(X)J) \subseteq \mathrm{C}^*(\pi, t)$ since

$$\theta_{\overline{\xi}a,\overline{\eta}} = t^n(\overline{\xi})\phi_0(a)t^m(\overline{\eta})^* = t^n(\overline{\xi})\left(\pi(a) - \psi_t(\phi(a))\right)t^m(\overline{\eta})^*,$$

for all $\overline{\xi} \in X^{\otimes n}, \overline{\eta} \in X^{\otimes m}$ and $a \in J$. Note that $\mathcal{K}(\mathcal{F}(X)J)$ is γ_z-invariant for all $z \in \mathbb{T}$, therefore $(q_J \circ \pi, q_J \circ t)$ inherits the gauge action $\{q_J \circ \gamma_z\}_{z\in\mathbb{T}}$. We will say that $(q_J \circ \pi, q_J \circ t)$ induces the *J-relative Fock representation.*

When $J \subseteq J_X := \ker \phi_X^{\perp} \cap \phi_X^{-1}(\mathcal{K}(X))$ then $(q_J \circ \pi, q_J \circ t)$ is isometric. This follows as in [10, Proposition 4.9]. In short, the $*$-homomorphism

$$\mathcal{K}(X^{\otimes n}J) \ni k \mapsto k \otimes \mathrm{id}_n \in \mathcal{L}(X^{\otimes n+1})$$

is injective since the restriction of ϕ_X on J is injective, and if

$$0 = \left\langle k\overline{\xi} \otimes \eta_1, k\overline{\xi} \otimes \eta_2\right\rangle_{X^{\otimes n+1}} = \left\langle \eta_1, \phi_X(\left\langle k\overline{\xi}, k\overline{\xi}\right\rangle_{X^{\otimes n}})\eta_2\right\rangle_X,$$

for all $\eta_1, \eta_2 \in X$ then the element $\left\langle k\overline{\xi}, k\overline{\xi}\right\rangle_{X^{\otimes n}}$ of J is also in $\ker \phi_X$. If $\pi(a) \in \mathcal{K}(\mathcal{F}(X)J)$ then $\phi_n(a) \in \mathcal{K}(X^{\otimes n}J)$ for all n, hence

$$\|a\|_A = \lim_n \|\phi_n(a)\| = \lim_n \|P_n\pi(a)P_n\| = 0,$$

since $\lim_n P_n k P_n = 0$ for all $k \in \mathcal{K}(\mathcal{F}(X))$, where we write P_n for the projection of $\mathcal{F}(X)$ onto the direct summand $X^{\otimes n}$.

One last property of $(q_J \circ \pi, q_J \circ t)$ is that $I'_{(q_J\circ\pi, q_J\circ t)} = J$. Indeed for $a \in I'_{(q_J\circ\pi, q_J\circ t)}$ let $k \in \mathcal{K}(X)$ such that $q_J \circ \pi(a) = \psi_{q_J\circ t}(k)$. Then injectivity of $q_J \circ \pi$ implies that $\phi_X(a) = k$ and

$$0 = q_J \circ \pi(a) - \psi_{q_J\circ t}(\phi_X(a)) = q_J(\pi(a) - \psi_t(\phi_X(a))) = q_J(\phi_0(a)),$$

thus $a \in J$. Conversely $J \subseteq I'_{(\pi,t)}$ by the J-covariance of (π, t).

The proof. Fix an ideal $J \subseteq J_X$. We will denote by (π_u, t_u) the universal representation of $\mathcal{O}(J, X)$. By the existence of the J-relative Fock representation we obtain that π_u is isometric. Furthermore, the universal property of $\mathcal{O}(J, X)$ provides the existence of a gauge action $\{\beta_z\}_{z\in\mathbb{T}}$ for (π_u, t_u). The cores in $\mathcal{O}(J, X)$ will be denoted by $\mathcal{B}_{[l,m]}$.

Suppose that (π, t) is a representation of X that admits a gauge action $\{\gamma_z\}_{z\in\mathbb{T}}$, that π is injective and that $I'_{(\pi,t)} = J$. Let $\Phi \colon \mathcal{O}(J, X) \to \mathrm{C}^*(\pi, t)$ be the canonical $*$-epimorphism. Then $\Phi \circ \beta_z = \gamma_z \circ \Phi$, and by a usual C*-argument it suffices to show that the restriction of Φ to the fixed point algebra $\mathcal{O}(J, X)^\beta$ is faithful. Since the fixed point algebra is the inductive limit of the cores $\mathcal{B}_{[0,N]}$ it suffices to show that the kernel of Φ intersects trivially all $\mathcal{B}_{[0,N]}$.

For $N=0$ we have that $\mathcal{B}_{[0,0]} = \pi_u(A)$ and by assumption $\Phi \circ \pi_u|_A = \pi$ is injective. For the inductive step let $N \geq 1$ be the least non-negative integer such that $\ker \Phi \cap \mathcal{B}_{[0,N]} \neq (0)$. Therefore $\ker \Phi \cap \mathcal{B}_{[0,N-1]} = (0)$ and let $f = \pi_u(a) + \sum_{n=1}^N \psi_{t_u^n}(k_n) \neq 0$ for $a \in A$ and $k_n \in \mathcal{K}(X^{\otimes n})$ such that $\Phi(f) = 0$. Then $\pi(a) = -\sum_{n=1}^N \psi_{t^n}(k_n)$. Let (e_i) be an approximate identity in $\mathcal{K}(X)$ and compute

$$\begin{aligned}\lim_i \psi_t(\phi_X(a)e_i) &= \lim_i \pi(a)\psi_t(e_i)\\ &= -\sum_{n=1}^N \lim_i \psi_{t^n}(k_n)\psi_t(e_i) = -\sum_{n=1}^N \psi_{t^n}(k_n),\end{aligned}$$

where we have used that $(\psi_t(e_i))$ acts as an approximate identity on all $\psi_{t^n}(\mathcal{K}(X^{\otimes n}))$. Therefore the net $(\psi_t(\phi_X(a)e_i))$ converges and so it is Cauchy in $\mathcal{B}(H)$. Thus so is the net $(\phi_X(a)e_i)$ in $\mathcal{K}(X)$ since π, and consequently ψ_t, is injective. Hence $(\phi_X(a)e_i)$ converges to some compact operator, say $k \in \mathcal{K}(X)$. Therefore

$$\pi(a) = -\sum_{n=1}^N \psi_{t^n}(k_n) = \lim_i \psi_t(\phi_X(a)e_i) = \psi_t(k).$$

Since $I'_{(\pi,t)} = J$ we obtain that $a \in J$. Consequently $\pi_u(a) = \psi_{t_u}(\phi_X(a)) \in \mathcal{B}_{[1,N]}$, which implies that $f \in \mathcal{B}_{[1,N]}$. However in this case

$$t_u(\eta)^* \cdot f \cdot t_u(\xi) = \sum_{n=1}^N t_u(\eta)^* \psi_{t_u^n}(k_n) t_u(\xi) \ \in \ \ker\Phi \cap \mathcal{B}_{[0,N-1]},$$

for all $\xi, \eta \in X$. By the choice of N we then obtain that $t_u(\eta)^* \cdot f \cdot t_u(\xi) = 0$. In particular $\psi_{t_u}(\theta_{\eta_1,\eta_2}) \cdot f \cdot \psi_{t_u}(\theta_{\xi_1,\xi_2}) = 0$, for all $\eta_1, \eta_2, \xi_1, \xi_2 \in X$, hence

$$\psi_{t_u}(\mathcal{K}(X)) \cdot f \cdot \psi_{t_u}(\mathcal{K}(X)) = (0).$$

But $\psi_{t_u}(\mathcal{K}(X))$ contains an approximate identity for $\mathcal{B}_{[1,N]}$, hence $f = 0$ which is a contradiction.

Conversely, suppose that (π, t) defines a faithful representation Φ of the relative $\mathcal{O}(J,X)$ such that $\Phi(\pi_u(a)) := \pi(a)$ and $\Phi(t_u(\xi)) = t(\xi)$. Then (π,t) admits a gauge action $\{\gamma_z\}_{z\in\mathbb{T}}$ by $\gamma_z := \Phi \circ \beta_z \circ \Phi^{-1}$, and $\pi = \Phi \circ \pi_u$ is injective. Finally we obtain that $I'_{(\pi,t)} = J$. Indeed, since $\pi_u(a) = \psi_{t_u}(\phi_X(a)) = \psi_{t_u}(k)$ for $a \in J$, then

$$\pi(a) = \Phi(\pi_u(a)) = \Phi(\psi_{t_u}(\phi_X(a))) = \Phi(\psi_{t_u}(k)) = \psi_t(k) = \psi_t(\phi_X(a)).$$

That is (π,t) is J-covariant, hence $J \subseteq I'_{(\pi,t)}$. Conversely if $\pi(a) = \psi_t(k)$ for some $k \in \mathcal{K}(X)$ then injectivity of π implies that $\phi_X(a) = k$ and

$$\pi_u(a) = \Phi^{-1}(\pi(a)) = \Phi^{-1}(\psi_t(k)) = \psi_{t_u}(\phi_X(a)).$$

In particular this holds for the injective J-relative Fock representation, thus $a \in J$, and the proof is complete.

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Department of Mathematics, Ben-Gurion University of the Negev, Be'er Sheva 84105, Israel

E-mail address: `kakariad@math.bgu.ac.il`